

FACTORISATION IN TOPOLOGICAL MONOIDS

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ABSTRACT. The aim of this paper is sketch a theory of divisibility and factorisation in topological monoids, where finite products are replaced by convergent products. The algebraic case can then be viewed as the special case of *discretely topologized* topological monoids.

In particular, we define the *topological factorisation monoid*, a generalisation of the factorisation monoid for algebraic monoids, and show that it is always *topologically factorial*: any element can be uniquely written as a convergent product of irreducible elements.

1. A PRIMER ON FACTORISATION IN DISCRETE MONOIDS

In this section, we give some basic definitions on the divisibility and factorisation theory of algebraic monoids. For additional information we refer to [5, 4]

An (algebraic) monoid H is a semi-group with a neutral element. In this paper, H is assumed to be abelian and cancellative. Unless otherwise stated, we write H multiplicatively and denote by $1 \in H$ its neutral element. However, the monoid $\mathbb{N} = (\mathbb{N}, +)$ will be written additively, with $0 \in \mathbb{N}$ the neutral element. We will also write the monoid \mathbb{N}^X (with X a set) additively, along with its sub-monoids.

We denote the set of units in H by H^\times , and say that H is *reduced* if $H^\times = \{1\}$. Since H^\times is a subgroup of H , we can form the factor monoid $H_{\text{red}} = H/H^\times$, which is reduced. More explicitly, $H_{\text{red}} = H/\sim$ where $a \sim b$ iff $a = eb$ for some $e \in H^\times$. By passing to H_{red} if necessary, we will in what follows assume that H is reduced. We denote the set of non-units by H^* , i.e. $H^* = H \setminus \{1\}$. Since H is assumed to be both cancellative and reduced, it follows that it is also torsion-free.

If $a, b \in H$ then we say that a divides b , and write $a|b$, if there exists a (necessarily unique) element $c \in H$ such that $ac = b$. An element $p \in H^*$ is said to be irreducible if $p = a_1 \cdots a_r$ with $a_i \in H$ implies that $p = a_j$ for some j . The irreducible elements in H are called atoms; we denote by $\mathcal{A}(H)$ the set of all atoms in H .

We say that $p \in H^*$ is prime if whenever p divides $a_1 \cdots a_r$, it divides some a_i . Note that a prime element is always irreducible.

A monoid H is *atomic* if $p \in H$ can be written as a finite product of atoms, and *factorial* if this factorisation is unique. In a factorial monoid, irreducible elements are prime. Hence unique factorisation into atoms implies unique factorisation into primes. A factorisation into

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primes is always unique, so if every element in H can be written as a product of primes, then H is factorial.

2. INFINITE PRODUCTS IN TOPOLOGICAL MONOIDS

We assume standard notions of topology and topological semigroups, as used for instance in [3, 6, 2].

In what follows, H will be a *topological monoid*, that is, a monoid with a topology on its underlying set such that the multiplication map $H \times H \rightarrow H$ is continuous. Following [3], we assume that the topology on H is Hausdorff. Note that any algebraic monoid is a topological monoid when endowed with the discrete topology. We also assume that H is abelian, cancellative and reduced.

To be able to talk about infinite products, we make the following definition (this is the same definition as is used in [2], but differently phrased):

Definition 2.1. A (possibly infinite) product

$$g = \prod_{j \in J} f_j, \quad f_j \in H, \quad J \text{ any set} \quad (1)$$

is *convergent* iff the net (ϕ, Δ) converges to g , where Δ is the directed set of all finite subsets of J , and for $F \in \Delta$, $\phi(F) = \prod_{j \in F} f_j$.

In detail: for every neighbourhood U of g , there is a finite subset $S \subset J$, such that for all finite subsets $S \subset T \subset J$ we have that $\prod_{j \in T} f_j \in U$.

Definition 2.2. We say that H allows arbitrary decimation if whenever $b = \prod_{s \in S} e_s$ is convergent, and $T \subset S$, then $\prod_{s \in T} e_s$ is convergent. We say that H allows finite decimation if $\prod_{s \in H} e_s$ is convergent whenever $S \setminus T$ is finite.

For instance, if H is complete, then H allows arbitrary decimation; this follows as in [2, III, §5.3, Proposition 3] (which treats the case of complete groups).

Example 2.3. Let \mathbb{R}^+ denote the monoid of non-negative real numbers, with the usual topology, and addition as the operation (which we'll write additively). Let \mathbb{Q}^+ denote the sub-monoid of non-negative rational numbers. Then \mathbb{Q}^+ is reduced, cancellative, Hausdorff, but not complete, and it does not allow arbitrary decimation. To see this, consider the sum

$$\sum_{k=1}^{\infty} 2^{-k} = 1, \quad (2)$$

which (in \mathbb{R}^+) can be decimated to yield any real number $x \in [0, 1]$.

Henceforth, we assume that H allows arbitrary decimation.

We can regard any infinite product of the form

$$U = \prod_{i \in I} g_i^{\alpha(i)}, \quad \alpha(i) \in \mathbb{N} \quad (3)$$

as an instance of (1) by taking $J = I$ and $f_j = g_i^{\alpha(i)}$, or by replacing $g^{\alpha(i)}$ with $\alpha(i)$ copies of g and enlarging the index set accordingly.

Lemma 2.4. Suppose that $x = \prod_{i \in I} e_i^{\alpha(i)}$ is a convergent product in H . Then if $\alpha(i) \geq \beta(i)$ for all $i \in I$ then the element $y = \prod_{i \in I} e_i^{\beta(i)}$ is a divisor of x .

Lemma 2.5. *If $x \in H^*$ then the sequence $(x^n)_{n=0}^\infty$ diverges.*

Proof. Suppose that $x^n \rightarrow a \in H$. Then $x \cdot x^n \rightarrow xa$, but we also have that $x \cdot x^n \rightarrow a$, hence $xa = a$. This is impossible since H is cancellative and $x \neq 1$. \square

Lemma 2.6. *Suppose that $x = \prod_{i \in I} e_i^{\alpha(i)}$ is a convergent product in H . Then all $\alpha(i) < \infty$, and there is no infinite subset $J \subset I$ such that $k, \ell \in J \implies e_k = e_\ell$.*

Proof. If $\prod_{i \in I} e_i^{\alpha(i)}$ is convergent, then so is all its sub-products. By Lemma 2.5, this means that no element can occur infinitely many times in the product. \square

Theorem 2.7. *Any convergent product $A = \prod_{j \in J} e_j$ can be expressed as*

$$A = \prod_{h \in H^*} h^{m(h)}, \quad m(h) = \# \{ j \in J \mid e_j = h \} < \infty. \quad (4)$$

The study of infinite products in H is thus reduced to the study of certain multisets on H . In the next section, we shall exploit a variant of this, when we consider multisets on the set of irreducible elements.

Definition 2.8. For $M \subset H$ we denote by $[M]^*$ the sub-monoid generated by all convergent products of elements in M .

Lemma 2.9. *If $M \subset H$, then*

$$M \subset [M] \subset [M]^* \subset \overline{[M]} \subset H \quad (5)$$

where $\overline{[M]}$ is the topological closure of $[M]$.

Proof. Note that the subspace $\overline{[M]}$ is a sub-monoid of H , since the closure of a sub-monoid is a sub-monoid.

The only non-trivial inclusion is $[M]^* \subset \overline{[M]}$. Let, as in (1), $U = \prod_{j \in J} f_j$ be a convergent product, with J any set; let Δ be the directed set of all finite subsets of J , let for $F \in \Delta$, $\phi(F) = \prod_{j \in F} f_j$, and let (ϕ, Δ) be the corresponding net. By definition of convergent products, the net converges to $U \in H$. Since $\phi(F) = \prod_{j \in F} f_j \in [M]$ for all $F \in \Delta$ it follows that $U \in \overline{[M]}$. \square

Example 2.10. Let

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{N}^+\} \subset [0, 1] \subset \mathbb{R}$$

be given the subspace topology, and let H be the following abelian topological monoid. As an algebraic monoid, H is the free abelian monoid on X ; we denote by e_0 the basis vector corresponding to 0, and by e_i the basis vector corresponding to $1/i$. There is a surjection

$$\begin{aligned} \phi : H &\rightarrow \mathbb{R} \\ e_0 &\mapsto 0 \\ e_i &\mapsto 1/i \quad \text{for } i > 0 \end{aligned}$$

We give H the smallest topology such that ϕ is continuous, that is, a sequence $f_v \rightarrow f$ in H iff $\phi(f_v) \rightarrow \phi(f)$ in \mathbb{R} . We will write the commutative monoid operation on H additively.

Let $M = \{e_n \mid n \in \mathbb{N}^+\} \subset H$. Since $1/n \rightarrow 0$ in \mathbb{R} , $e_n \rightarrow e_0$ in H , hence $e_0 \in \overline{[M]}$. We claim that $e_0 \notin [M]^*$. To see this, let $g : X \rightarrow \mathbb{R}$ be the natural inclusion, which is of course continuous and closed. Suppose that $e_0 = \sum_{i \in I} e_{c_i}$ with $c_i \in \mathbb{N}^+$ is a convergent sum, then by continuity of g we get that $0 = g(e_0) = g(\sum_{i \in I} e_{c_i}) = \sum_{i \in I} g(e_{c_i})$. But $g(e_{c_i}) > 0$ in the natural total order on \mathbb{R} , hence $\sum_{i \in I} g(e_{c_i}) > 0$, a contradiction.

Definition 2.11. We say that the topological monoid H is *almost discrete* if all convergent products in H are finite, that is, for all $M \subset H$, we have that $[M] = [M]^*$.

Example 2.12. Let M be the multiplicative monoid of $K[[x_1, \dots, x_n]]$, with the \mathbf{m} -adic topology, and let $H = \{1\} \cup f \in Mf(0) = 0$. Then H is almost discrete (any convergent infinite product of elements in $K[[x_1, \dots, x_n]]$ will converge to $0 \notin H$, but not discrete (every non-polynomial in H is the limit of polynomials).

3. FACTORISATION: ATOMS, PRIMES, AND THEIR TOPOLOGICAL COUNTERPARTS

Definition 3.1. We say that $p \in H$ is *topologically prime* if whenever p divides a convergent product, it divides some factor. We denote by $\mathcal{B}(H)$ the set of topologically prime elements in H .

Lemma 3.2. Call $a \in H^*$ topologically irreducible if it can not be written as a convergent product of elements, all different from a . Then a is topologically irreducible if and only if it is irreducible.

Proof. Since H is reduced, a is irreducible if and only if it can not be written as a finite product of non-units, all different from a . Thus if a is topologically irreducible, it is irreducible.

For the converse, we need to use that H allows finite decimation. Suppose that a is irreducible, and that $a = \prod_{i \in I} e_i$. Let $j \in I$. Then $a = e_j b$, where $b = \prod_{i \in I \setminus \{j\}} e_i$. Since a is irreducible, either $a = e_j$ or $a = b$. If $a = e_j$, we are done. If $a = b$, then $a = e_j b = b$, which is impossible since H is cancellative. \square

Definition 3.3. Suppose that

$$f = \prod_{i \in I} a_i = \prod_{j \in J} b_j \quad (6)$$

are two convergent factorisations of f into non-units. We say that these factorisations are equivalent if

$$\forall h \in H^* : \quad \# \{i \in I \mid a_i = h\} = \# \{i \in I \mid a_i = h\} \quad (7)$$

Definition 3.4. H is *topologically [atomic, prime atomic]* if every non-unit h can be written as a convergent product of [atoms, topologically prime elements]. It is *topologically [factorial, prime factorial]* if any two such factorisations of h are equivalent.

Proposition 3.5. Suppose that H is topologically prime atomic, and let $x \in H^*$. Then the following are equivalent:

- (i) x is topologically prime,
- (ii) x is prime,
- (iii) x is irreducible.

Proof. It suffices to show that an irreducible element is topologically prime, so suppose that x is irreducible. Since H is topologically prime atomic, x may be written as a convergent product of topologically prime elements. We claim that this product must have only one factor. Hence, x is topologically prime.

To establish the claim, we argue by contradiction, and write $x = \prod_{i \in I} e_i$ with e_i topologically prime, and where I is not a singleton. Choose an $j \in I$ and put $y = e_j$ and $z = \prod_{i \in I \setminus \{j\}} e_i$. Since H allows finite decimation, the latter product is convergent. Hence $x = yz$, in contradiction to the fact that x is irreducible. \square

Proposition 3.6. *Suppose that H is topologically factorial. Then atoms in H are prime.*

Proof. Suppose that x is an atom in H , and that $x \mid ab$, with $a, b \in H$. Then there exists $y \in H$ with $xy = ab$. Since H is topologically factorial, we can uniquely factor a, b, y into atoms:

$$a = \prod_{i \in I} u_i, \quad b = \prod_{j \in J} v_j, \quad y = \prod_{k \in K} w_k.$$

We can assume that I, J, K are pair-wise disjoint. Applying [2, III, §5.3, Proposition 3]¹ we have that

$$x \prod_{k \in K} w_k = \prod_{i \in I} u_i \times \prod_{k \in K} w_k = \prod_{\ell \in I \cup J} z_\ell, \quad z_\ell = \begin{cases} u_i & \text{if } \ell = i \in I, \\ v_j & \text{if } \ell = j \in J. \end{cases}$$

Since H is topologically factorial, factorisation into atoms is unique, hence $x = z_\ell$ for some $\ell \in I \cup J$. Without loss of generality, assume that $\ell = i \in I$, so that $x = u_i$. Then the fact that H allows finite decimation implies that $x \mid a$. \square

Definition 3.7. H *allows dissociation* if whenever $b = \prod_{s \in S} e_s$ is convergent, and for each $s \in S$, $e_s = \prod_{l \in G_s} f_l$ is convergent, then $b = \prod_{l \in G} f_l$ is convergent, where G is the disjoint union of the G_s 's.

Example 3.8. There are lots of non-reduced, but cancellative and even complete monoids which do not allow expansion. For instance, in the additive group of the reals, if we put $e_k = k + (-k) = 0$ we have that $0 = \sum_{k \in \mathbb{Z}^+} (k + (-k))$, but $\sum_{k \in \mathbb{Z}} k$ is not convergent.

Example 3.9. Consider the monoid \mathbb{Q}^+ of Example 2.3. We claim that this monoid does allow dissociation, but does not allow us to perform the reordering $\sum_{i \in I} \sum_{j \in J} e_{ij} = \sum_{j \in J} \sum_{i \in I} e_{ij}$. To establish the first claim, we note that in \mathbb{R}^+ all summable nets are countable, thus we need only to consider sequences. Furthermore, since everything is positive, all convergent sums in \mathbb{R}^+ are absolutely convergent. Thus, if $\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = A$, then $\sum_{(i,j)} a_{ij} = A$. If the first sum has all summands in \mathbb{Q}^+ and converges to a rational value, then so does the second sum.

However, the ‘‘column sums’’ $\sum_{i=1}^{\infty} a_{ij}$ need not be rational.

We believe that our assumptions on H (cancellative, reduced) are *not* enough to guarantee that H allows dissociation, thus we postulate this in the next proposition.

¹That Proposition deals with topological groups, but the proof works verbatim for topological monoids.

Proposition 3.10. *Suppose that H is topologically factorial and allows dissociation. Then atoms in H are topologically prime.*

Proof. Suppose that x is an atom in H . By the previous proposition, x is prime. Suppose that $x \mid \prod_{i \in I} f_i$, with $f_i \in H$, and write each f_i as a convergent product $f_i = \prod_{a \in \mathcal{A}(H)} a^{g_i(a)}$.

$$\prod_{i \in I} f_i = \prod_{i \in I} \prod_{a \in \mathcal{A}(H)} a^{g_i(a)},$$

but on the other hand, $\prod_{i \in I} f_i = xb$ for some $b \in H$. Write $b = \prod_{a \in \mathcal{A}(H)} a^{h(a)}$, then

$$\prod_{i \in I} \prod_{a \in \mathcal{A}(H)} a^{g_i(a)} = x \prod_{a \in \mathcal{A}(H)} a^{h(a)}.$$

Since H allows dissociation, we get that

$$x \prod_{a \in \mathcal{A}(H)} a^{h(a)} = \prod_{(i,a) \in I \times \mathcal{A}(H)} a^{g_i(a)},$$

and since H is topologically factorial, factorisation into atoms is unique, hence x occurs in the right hand side. \square

Proposition 3.11. *If $b = \prod_{s \in S} e_s^{\alpha_s}$ is a convergent product in H of topologically prime elements, then α_s is the maximal integer $r \geq 0$ such that $e_s^r \mid b$. Hence, the factorisation of an element into a convergent product of topologically prime elements, if it exists, is unique.*

Proof. For any topologically prime element $p \in M$, we have that $p \mid b$ iff $p = e_s$ for some $s \in S$. To see this, first note that if $p = e_s$ then $b = e_s(e_s^{\alpha_s-1} \prod_{t \in S \setminus \{s\}} e_t^{\alpha_t})$. The right hand side is a product of e_s and a convergent since H allows finite decimation.

Conversely, if $p \mid b$ then by definition of topologically primeness there is an $s \in S$ such that $p \mid e_s^{\alpha_s}$. Applying the property of topologically primeness again, we get that $p \mid e_s$. Clearly, different topologically prime elements do not divide each other, hence $p \mid e_s$ iff $p = e_s$. It follows that α_s is the maximal integer r such that $e_s^r \mid b$. \square

Corollary 3.12. *A topologically prime atomic monoid is topologically prime factorial.*

So, we have the following implications:

topologically prime atomic \iff topologically prime factorial \implies topologically factorial.

The last implication can be reversed if H allows dissociation.

4. THE TOPOLOGICAL FACTORISATION HOMOMORPHISM

Recall that the free abelian monoid $\mathcal{F}(\mathcal{A}(H))$ is called the *factorisation monoid* of H , and the canonical homomorphism

$$\pi_H : \mathcal{F}(\mathcal{A}(H)) \rightarrow H \tag{8}$$

is called the *factorisation homomorphism*. If $p \in H$, then the elements in $\pi_H^{-1}(p)$ are called the *factorisations* of p . This homomorphism gives a lot of information about the factorisation properties of H : we have that H is atomic iff π_H is surjective, and factorial iff π_H is bijective.

We now make a construction which captures also the infinite factorisations. First, we introduce some notation for the topological monoid \mathbb{N}^X , the set of all functions $X \rightarrow \mathbb{N}$,

where X is any set. This is a topological monoid with the operation of point-wise addition (we'll write the operation additively), and the topology of point-wise convergence. It is also a partially ordered set with point-wise comparison.

Definition 4.1. For $x \in X$, we define

$$\chi_x : X \rightarrow \mathbb{N}$$

$$\chi_x(y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$$

Thus any $f : X \rightarrow \mathbb{N}$ may be written as a convergent sum $f = \sum_{x \in X} f(x)\chi_x$.

Definition 4.2. The partially defined map

$$\overline{\pi_H} : \mathbb{N}^{\mathcal{A}(H)} \rightarrow H \quad (9)$$

$$\sum_{a \in \mathcal{A}(H)} v(a)\chi_a \mapsto \prod_{a \in \mathcal{A}(H)} a^{v(a)} \quad (10)$$

is defined whenever the right-hand side of (10) is a convergent product. Denote by $Z(H) \subset \mathbb{N}^{\mathcal{A}(H)}$ the domain of definition of $\overline{\pi_H}$. We call $Z(H)$ *topological factorisation monoid* of H ; note that it contains the factorisation monoid of H , since the latter corresponds to the finitely supported maps $\mathcal{A}(H) \rightarrow \mathbb{N}$. In what follows, we regard $\overline{\pi_H}$ as a map $Z(H) \rightarrow H$ and call it the *topological factorisation homomorphism*.

If $p \in H$, then $\overline{\pi_H}^{-1}(p)$ is the set of (topological) factorisations of p .

Clearly, H is topologically atomic iff $\overline{\pi_H}$ is surjective, and topologically factorial iff $\overline{\pi_H}$ is bijective.

For each $a \in \mathcal{A}(H)$, the projection map $\text{pr}_a : Z(H) \rightarrow \mathbb{N}$ is defined by $(e_v)_{v \in \mathcal{A}(H)} \mapsto e_a$. We topologize $Z(H)$ by giving it the initial topology with respect to $\overline{\pi_H}$ and all the projection maps pr_a , where \mathbb{N} is discretely topologized.

Thus, $Z(H)$ has the weakest topology such that $\overline{\pi_H}$ and all the projections pr_a are continuous, and a net f_i converges to f in $Z(H)$ iff $\overline{\pi_H}(f_i) \rightarrow \overline{\pi_H}(f)$ and $\forall a \in \mathcal{A} : \text{pr}_a(f_i) \rightarrow \text{pr}_a(f)$. It is easy to see that this topology is Hausdorff.

Lemma 4.3. *With respect to the component-wise partial order on $\mathbb{N}^{\mathcal{A}(H)}$, $Z(H)$ is an order ideal, i.e. if $c, d : \mathcal{A}(H) \rightarrow \mathbb{N}$, $c \in Z(H)$, and $d(a) \leq c(a)$ for all $a \in \mathcal{A}(H)$, then $d \in Z(H)$.*

Proof. This is a direct consequence of our assumption that H allows arbitrary decimation. \square

Proposition 4.4. *$Z(H)$ is a topological monoid, and $\overline{\pi_H}$ is an homomorphism of topological monoids.*

Proof. To show that $Z(H)$ is a algebraic monoid, we must show that if $f, g \in Z(H)$ then $h = f + g \in \mathbb{N}^{\mathcal{A}(H)}$ is in fact in $Z(H)$. So, we must show that $\prod_{a \in \mathcal{A}(H)} a^{h(a)}$ is convergent. Let W be a neighbourhood of $\overline{\pi_H}(h) \in H$. Since multiplication in H is continuous, there is a neighbourhood U of $\overline{\pi_H}(f)$ and a neighbourhood V of $\overline{\pi_H}(g)$ such that $UV \subset W$. Since $\prod_{a \in \mathcal{A}(H)} a^{f(a)}$ and $\prod_{a \in \mathcal{A}(H)} a^{g(a)}$ are convergent, there is a finite $S \subset \mathcal{A}(H)$ such that for any finite subset $T \subset \mathcal{A}(H)$ we have that $\prod_{a \in T} a^{f(a)} \in U$ and $\prod_{a \in T} a^{g(a)} \in V$, hence $\prod_{a \in T} a^{h(a)} \in UV \subset W$. Thus, $\prod_{a \in \mathcal{A}(H)} a^{h(a)}$ is convergent, showing that $Z(H)$ is an algebraic

monoid. The product obviously converges to $\overline{\pi_H}(h)$, so $\overline{\pi_H}$ is a homomorphism of algebraic monoids. By definition, $\overline{\pi_H}$ is continuous.

It remains to see that addition in $Z(H)$ is continuous. Let $f_i \rightarrow f$, $g_i \rightarrow g$ in $Z(H)$, then by definition $\overline{\pi_H}(f_i) \rightarrow \overline{\pi_H}(f)$ and $\text{pr}_a(f_i) \rightarrow \text{pr}_a(f)$, and likewise for g . Since H is a topological monoid, $\overline{\pi_H}(f_i + g_i) = \overline{\pi_H}(f_i)\overline{\pi_H}(g_i) \rightarrow \overline{\pi_H}(f)\overline{\pi_H}(g) = \overline{\pi_H}(f + g)$, and similarly $\text{pr}_a(f_i + g_i) \rightarrow \text{pr}_a(f + g)$. Thus $f_i + g_i \rightarrow f + g$. \square

Lemma 4.5. *$Z(H)$ is reduced, cancellative, and allows arbitrary decimation.*

Proof. Since \mathbb{N}^X is reduced and cancellative for all X , we have that $Z(H)$ is a submonoid of a reduced, cancellative monoid, and hence it is reduced and cancellative.

Let $\sum_{i \in I} g_i = h$ be a convergent sum in $Z(H)$, and let $J \subset I$. We want to show that $\sum_{i \in J} g_i$ is convergent, i.e. that the following two conditions hold:

1. $\prod_{i \in J} \overline{\pi_H}(g_i)$ converges,
2. for all $a \in \mathcal{A}H$, $\sum_{i \in J} g_i(a) < \infty$.

The first property follows since we know that $\prod_{i \in I} \overline{\pi_H}(g_i)$ converges, and that H allows arbitrary decimation. The second property follows since we know that $\sum_{i \in I} g_i(a) < \infty$. \square

We now show that this definition generalises the discrete one.

Theorem 4.6. *If H is discrete, then so is $Z(H)$.*

Proof. The fact that H is discrete means that $Z(H)$ consists precisely of the finitely supported maps $\mathcal{A}(H) \rightarrow H$. Suppose that $c_i \rightarrow c \in Z(H)$. We put $x = \overline{\pi_H}(c)$, and note that since $\overline{\pi_H}(c_i) \rightarrow \overline{\pi_H}(c) = x$, and since H is discrete, there is an N_1 such that $\overline{\pi_H}(c_i) = x$ for all $i > N_1$.

We have that c is supported on a finite set $A \subset \mathcal{A}(H)$, and for each $a \in A$, there is an M_a such that $\text{pr}_a(c_i) = \text{pr}_a(c)$ whenever $i > M_a$. Thus there is an N_2 such that $\text{pr}_a(c_i) = \text{pr}_a(c)$ whenever $i > N_2$ and $a \in A$.

Suppose now that $i > N_1, N_2$. Then

$$\overline{\pi_H}(c_i) = \prod_{a \in A} a^{\text{pr}_a(c_i)} \prod_{b \in \mathcal{A}(H) \setminus A} b^{\text{pr}_b(c_i)} = \overline{\pi_H}(c) = \prod_{a \in A} a^{\text{pr}_a(c)} \prod_{b \in \mathcal{A}(H) \setminus A} b^{\text{pr}_b(c)} = \prod_{a \in A} a^{\text{pr}_a(c_i)},$$

so

$$\prod_{b \in \mathcal{A}(H) \setminus A} b^{\text{pr}_b(c_i)} = 1,$$

which means that $\text{pr}_b(c_i) = 0$ for all $b \notin A$. Hence $c_i = c$ when $i > N_1, N_2$, so all convergent nets are stationary after a finite number of steps, which means that $Z(H)$ has the discrete topology. \square

Lemma 4.7. *The atoms (and the topologically prime elements) of $\mathbb{Z}(H)$ are precisely the elements $\{\chi_a | a \in \mathcal{A}(H)\}$*

Proof. If $a \in \mathcal{A}(H)$ then $\chi_a \in Z(H) \subset \mathbb{N}^{\mathcal{A}(H)}$. Since χ_a is irreducible in $\mathbb{N}^{\mathcal{A}(H)}$, it is irreducible in $\mathbb{Z}(H)$. If $f \in Z(H)$, then $f = \sum_{a \in \mathcal{A}(H)} f(a)\chi_a$, so if $f(a), f(b) \neq 0$ for $a \neq b$, then f is not topologically irreducible, hence (Lemma 3.2) not irreducible. We have thus shown that $\mathcal{A}(Z(H)) = \{\chi_a | a \in \mathcal{A}(H)\}$.

By Lemma 4.5 we have that all χ_a are topologically prime. A topologically prime element is prime, hence irreducible, hence of the form χ_a . \square

Lemma 4.8. *If H is a topologically factorial, and if $H \ni f = \prod_{a \in \mathcal{A}(H)} a^{v(a)}$, $H \ni g = \prod_{a \in \mathcal{A}(H)} a^{w(a)}$, then $f \mid g$ iff $\forall a \in \mathcal{A}(H) : v(a) \leq w(a)$.*

Proof. The underlying algebraic monoid of H is isomorphic to $Z(H) \subset \mathbb{N}^{\mathcal{A}(H)}$, and $Z(H)$ is an order ideal. \square

Example 4.9. For a topologically factorial monoid the topological factorisation homomorphism is an isomorphism of algebraic monoids, but it need not be an isomorphism of topological monoids. As an example, if H is the topological monoid of Example 2.10 then the factorisation homomorphism $\overline{\pi_H}$ maps a_0 to e_0 and a_n to e_n . Now, consider the sequence $f_i = a_i$ for $i > 1$. We have that $\overline{\pi_H}(f_i) \rightarrow e_0 = \overline{\pi_H}(f_0)$. However, $\text{pr}_{a_0}(f_i) = 0$ for all $i > 0$, so $\text{pr}_{a_0}(f_i) \rightarrow 0$, whereas $\text{pr}_{a_0}(f_0) = 1$. Thus, the inverse is not continuous.

Recall that for a discretely topologized monoid, the factorisation monoid $\mathcal{F}(\mathcal{A}(H))$ is free abelian. Similarly:

Theorem 4.10. *$Z(H)$ is topologically prime factorial.*

Proof. Each element $f \in Z(H)$ can be written uniquely as $f = \sum_{a \in \mathcal{A}(H)} \chi_a^{f(a)}$; this sum is convergent with respect to the topology of point-wise convergence, and by construction, its image under $\overline{\pi_H}$ is also convergent. \square

Moreover:

Theorem 4.11. *$Z(Z(H)) \simeq Z(H)$ as topological monoids.*

Proof. We define

$$\begin{aligned} \Xi : Z(H) &\rightarrow \mathbb{N}^{\mathcal{A}(Z(H))} \\ \sum_{a \in \mathcal{A}(H)} f(a)a &\mapsto \sum_{a \in \mathcal{A}(H)} f(a)\chi_a \end{aligned} \tag{11}$$

where we have used the fact that $\mathcal{A}(Z(H)) = \{ \chi_a \mid a \in \mathcal{A}(H) \}$ (Lemma 4.7). This map is obviously an injective homomorphism of algebraic monoids. Since

$$\sum_{a \in \mathcal{A}(H)} f(a)\chi_a \in Z(Z(H))$$

if and only if

$$\sum_{a \in \mathcal{A}(H)} f(a)a \text{ converges in } Z(H)$$

if and only if

$$\prod_{a \in \mathcal{A}(H)} a^{f(a)} \text{ converges in } H$$

if and only if

$$\sum_{a \in \mathcal{A}(H)} f(a)a \in Z(H)$$

we get that the image of Ξ is exactly $Z(Z(H))$, thus that $Z(H) \simeq Z(Z(H))$ as algebraic monoids. Henceforth, we regard Ξ as a map $\Xi : Z(H) \rightarrow Z(Z(H))$. A simple calculation

shows that it is a section to the topological factorisation homomorphism $\overline{\pi_{Z(H)}} : Z(Z(H)) \rightarrow Z(H)$, i.e. that $\overline{\pi_{Z(H)}} \circ \Xi : Z(H) \rightarrow Z(H)$ is the identity.

It remains to show that Ξ is continuous, with continuous inverse. Let $(g_v)_{v \in V}$ be a net in $Z(H)$, and let $g \in Z(H)$, with

$$g = \sum_{a \in \mathcal{A}(H)} r(a)a, \quad g_v = \sum_{a \in \mathcal{A}(H)} r_v(a)a \quad (12)$$

Then

$$g_v \rightarrow g \text{ in } Z(H)$$

if and only if

$$\overline{\pi_H}(g_v) \rightarrow \overline{\pi_H}(g) \text{ in } H \text{ and } \forall b \in \mathcal{A}(H) : \text{pr}_b(g_v) \rightarrow \text{pr}_b(g)$$

if and only if

$$\overline{\pi_H} \left(\sum_{a \in \mathcal{A}(H)} r_v(a)a \right) \rightarrow \overline{\pi_H} \left(\sum_{a \in \mathcal{A}(H)} r(a)a \right) \text{ in } H$$

and

$$\forall b \in \mathcal{A}(H) : \text{pr}_b \left(\sum_{a \in \mathcal{A}(H)} r_v(a)a \right) \rightarrow \text{pr}_b \left(\sum_{a \in \mathcal{A}(H)} r(a)a \right)$$

which holds if and only if

$$\prod_{a \in \mathcal{A}(H)} a^{r_v(a)} \rightarrow \prod_{a \in \mathcal{A}(H)} a^{r(a)} \text{ and } \forall b \in \mathcal{A}(H) : r_v(b) \rightarrow r(b).$$

On the other hand,

$$\Xi(g_v) \rightarrow \Xi(g) \text{ in } Z(Z(H))$$

if and only if

$$\overline{\pi_{Z(H)}}(\Xi(g_v)) \rightarrow \overline{\pi_{Z(H)}}(\Xi(g)) \text{ in } Z(H) \text{ and } \forall b \in \mathcal{A}(Z(H)) : \text{pr}_b(\Xi(g_v)) \rightarrow \text{pr}_b(\Xi(g)),$$

if and only if

$$g_v \rightarrow g \text{ in } Z(H) \text{ and } \forall b \in \mathcal{A}(H) : r_v(b) \rightarrow r(b)$$

Hence,

$$g_v \rightarrow g \iff \Xi(g_v) \rightarrow \Xi(g),$$

so Ξ is continuous, with continuous inverse. □

5. THE CASE OF RESTRICTED DECIMATION — A COUNTEREXAMPLE

If H does not allow arbitrary decimation, then few of the previous results hold. We'll explore this in an example.

Consider the submonoid $H \subset \mathbb{N}^{\mathbb{N}}$ consisting of those functions $\mathbb{N} \rightarrow \mathbb{N}$ which are either finitely supported, or ≥ 1 everywhere. Then, the functions χ_j , with $j \in \mathbb{N}$, are topologically prime, and topologically irreducible. Now consider the function f which is constantly 1. Note that f is irreducible, since it can not be decomposed as a non-trivial finite product. In fact, f is prime: if f divides $\prod_{j=1}^n g_j$ then at least one g_j must be ≥ 1 everywhere, and then f divides that g_j . However, f can be written as a convergent (infinite) product of atoms, so it is neither topologically irreducible, nor topologically prime.

Note that $f = \prod_{i=0}^{\infty} \chi_i$, yet χ_0 is not a divisor of f , since

$$\frac{f}{\chi_0} = \prod_{i=1}^{\infty} \chi_i \notin H.$$

So $Z(H)$ is not an order ideal. This is true even if we instead define the topological factorisation monoid as a set of maps from the topologically irreducible elements to \mathbb{N} .

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